

CONSIDERING THE PASADENA “PARADOX”

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Abstract

Nover and Hájek (2004) suggested a variant of the St Petersburg game which they dubbed the Pasadena game. They hold that their game “is more paradoxical than the St Petersburg game in several aspects”. The purpose of this article is to demonstrate theoretically and to validate by simulation, that their game does not lead to a paradox at all, let alone in the St Petersburg game sense. Their game does not produce inconsistencies in decision theory.

Keywords: expected values; St Petersburg paradox; decision rules; simulation; harmonic series

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The paradox of the St Petersburg paradox

The St Petersburg game and the associated paradox are important in the field of decision theory involving situations of risk and uncertainty (Vivian, 2003). Any variant of this game is of interest. Nover and Hájek (2004) suggest such a variant which they dubbed the Pasadena game, which they hold leads to a paradox more profound than the St Petersburg paradox. To accept that the Pasadena game produces a paradox, in the St Petersburg game (or decision theory) sense, one must first establish what is meant by a *paradox* in that sense.

In the 1700s it was thought that people should make decisions in terms of the mathematical expectancy decision criterion, a decision criterion usually credited to Pierre de Fermat (1601-1665) and Blaisé Pascal (1623-1662). Daniel Bernoulli (1738/1954) tried to demonstrate that decision makers do not, in fact, make decisions in line with expected values. He gave a number of examples which he thought illustrated that point. One of these is what is known as the St Petersburg game. In this game a fair coin is flipped until a head appears,

whereupon Peter (the casino owner) pays Paul (the gambler) $\$2^{i-1}$, if the head appeared on the i^{th} flip. According to traditional wisdom¹ the expected value is, $E\{X\} = \sum p_i C_i$ where p_i is the probability that outcome C_i will appear. Accordingly:

$$E\{X\} = 2^0/2^1 + 2^1/2^2 + 2^2/2^3 \dots$$

$$E\{X\} = 1/2 + 1/2 + 1/2 \dots = 8$$

Thus according to traditional wisdom, if the $E\{X\}$ decision criterion is applied to the St Petersburg game, one would expect gamblers to be prepared to wager very very large sums to play the game. Empirical evidence, on the other hand, indicates that gamblers, quite rightly, are not willing to wager more than a few dollars. Herein lies the paradox or as Todhunter (1865: 220) put it, “The paradox then is that the mathematical theory is apparently directly opposed to the dictates of common sense.” In order that the Pasadena game constitute a paradox, in the St Petersburg game sense, it is necessary to show that empirical evidence about amounts wagered is grossly out of line with that suggested by decision theory, in particular, the expected value decision criterion.

With this background the game suggested by Nover and Hájek (2004) and dubbed by them to be the Pasadena game, is examined. In the Pasadena game, the prize is $\$((-1)^{i-1} \cdot 2^i)/i$

which can be compared to 2^{i-1} of the St Petersburg game. The Pasadena prize is much more complex and can result in the gambler, paying the casino, instead of being paid by the casino. The $E\{X\}$ of the Pasadena game is determined by the following harmonic series:

$$\begin{aligned} E\{X\} &= 2^{-1} \cdot \$((-1)^{1-1} 2^1)/1 + 2^{-2} \cdot \$((-1)^{2-1} 2^2)/2 + \dots \$((-1)^{i-1}/i + \dots \\ E\{X\} &= 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 \dots \\ &\$((-1)^{i-1}/i + \dots \end{aligned} \quad (1)$$

This is the series for $\ln 2$. Thus:

$$E\{X\} = \ln 2 \approx \$0.69$$

So unlike the St Petersburg game, with an infinite expected value, the Pasadena game has a very modest, finite expected value. Nover and Hájek (2004) unfortunately do not suggest how much they believe a gambler would be prepared to pay to play their game, but if it is shown that gamblers are prepared to pay an amount of the order of \$0.69 then there is of course, no paradox in the St Petersburg game or any other sense. Since the expected value of their game is a very modest amount, it seems reasonable to accept that gamblers will be prepared to play an amount of this order. Thus unlike the St Petersburg game, the Pasadena game does not on the face of it produce a paradox, at all, in the field of decision theory.

If it is clear that the Pasadena game does not produce a paradox why then do the authors believe their game produces a problem? It is because they hold that the expected value of their game depends on how the terms in the series are arranged and summated. This in itself is an extraordinary view, since it is contrary to the well-known commutative law of addition $A+B = B+A$. One should not expect the mere rearrangement of a series to produce different results.

They liken the series to a pack of cards which falls on the ground and when picked up can appear in different sequences, which can then be summated and depending on the way that the cards are rearranged, a different outcome arises. They hold that if the series is arranged as follows:

$$E\{X\} = 1 - 1/2 + 1/3 - 1/4 \dots \$((-1)^{i-1}/i + \dots \quad (2)$$

then $E\{X\} = \ln 2$

However, if the series is rearranged as follows, they hold:

$$\begin{aligned} E\{X\} &= 1 + (-1/2 - 1/4 - 1/6 - 1/8 - 1/10) + \\ &\quad 1/3 + (-1/12 - 1/14 - 1/16 - 1/18 - \\ &\quad 1/20) \dots \\ &= 1/5 + (-1/22 \dots) \\ &= \ln 2 + 1/2 \ln (1/5) \\ &= -0.11 \end{aligned}$$

However, if the series is rearranged as follows, they hold:

$$\begin{aligned} E\{X\} &= 1 + (1/3 + 1/5 \dots 1/41 - 1/2) + \\ &\quad (1/43 + \dots + 1/511 - 1/4) + \\ &\quad (1/513 + \dots + 1/5279 - 1/6) + \dots \\ &= 1 + 1.00406 + 1.00028 + \\ &\quad 1.000011 + \dots \quad (3) \\ &= \infty \end{aligned}$$

However, if the series is rearranged as follows, they hold:

$$\begin{aligned} E\{X\} &= (1 - 1/2 - 1/4 - 1/6 - 1/8 - \dots - 1/62) + \\ &\quad (1/3 - 1/64 \dots 1/906) + \\ &\quad (1/5 - 1/908 \dots - 1/9998) + \\ &\quad \dots \quad (4) \\ &\approx -1 - 1 - 1 \dots \\ &= -\infty \end{aligned}$$

And thus according to them, simply by rearranging the series, the $E\{X\}$ can have any value from $-\infty$ to $+\infty$. According to this view the $E\{X\}$ is indeterminable.

Even if this view is correct (which is doubted), this still does not produce a paradox in the St Petersburg paradox or decision theory sense. If the expected value cannot be determined, then it cannot be applied to make a decision and no paradox arises. The problem and its solution then lies in the theory of harmonic series, not decision theory. If their view is correct, it means that the harmonic series which is thought to sum to $\ln 2$, in fact does not do so and as such will not do so in any application of the series and not only in the field of decision theory.

Notwithstanding this, it can be shown that the Pasadena game does not produce a problem for decision theory since the series is (a) infinite in length only if an infinite number of games is played (which is impossible), (b) the order in which if the terms arise, arise naturally and once the vast number of negative terms, omitted by the authors in series (3) or positive terms

omitted in series (4) is accounted for, the expected value is finite and still converges on $\ln 2$.

2

Bernoulli's methodology after correcting his error

The methodology suggested by Vivian (2003) and demonstrated by simulation Vivian (2004) can be applied to the Pasadena game. The methodology is essentially the same as suggested by Bernoulli (1954/1738) himself after correcting for some errors in his methodology. Bernoulli (1954/1738: 32) pointed out that if the game is played N times, half of these cases are expected to end at the first flip of the coin, a quarter at the second, an eighth at the third and so on to infinity (*sic*)².

Thus if the game is played N times, this series can be expressed as follows:

$$N = N/2 + N/4 + N/8 \dots \infty \text{ (sic)}$$

or

$$N = n_1 + n_2 + n_3 + n_4 + \dots \infty \text{ (sic) Where } n_1 = N/2 \text{ and } n_2 = N/4 \text{ etc}$$

where:

$$N = \sum n_i$$

$$\text{and } 1 = \sum 1/2 + 1/4 \dots$$

$$1 = \sum p_i$$

Where p_i is the probability of the i^{th} term.

Karl Menger (in Bernoulli 1954/1738: 32 note 10) pointed to the error in Bernoulli's statement when he wrote "Since the number of cases $[N]$ is infinite, it is impossible to speak about half of the cases, one quarter of the cases, etc and the letter N in Bernoulli's argument is meaningless." N simply cannot equal ∞ if subjected to mathematical manipulation.

There are other errors in his statement. In practice not exactly $N/2$ games will terminate at the first throw of the coin. For example if a coin is tossed 10 times, it is expected that in 5 cases a "head" will appear, but it is possible that any number from 0 to 10 will in fact appear. Thus a more accurate statement is:

$N = (N/2 \pm \alpha_1) + (N/4 \pm \alpha_2) + (N/8 \pm \alpha_3) + \dots$
Where $\sum \alpha_i = 0$ and $\alpha_1/N, \alpha_2/N, \alpha_3/N, \alpha_4/N, \alpha_i/N \dots$ all, as a consequence of the Law of Large Numbers, tend to 0 as N , the number of games played tend to infinity. There is thus an implicit assumption in the traditional solution to St Petersburg that an infinite number of games is played, a factual impossibility. What is more important in practice, is what happens when a finite number of games is played, since this is possible.

Bernoulli's error can be corrected by setting N to a finite number, say, $N=2^k$. This produces a finite series of terms as indicated in Table 1, which indicates the number of games expected to end after each flip of the coin. There is an assumption that N is large and hence each $\alpha_i/N = 0$.

Table 1

Expected number of games terminating at each flip, if the game is played 2^k times

Term	T_1	T_2	T_3	...	T_k
n_i	2^{k-1}	2^{k-2}	2^{k-3}	...	2^{k-k}
p_i	2^{-1}	2^{-2}	2^{-3}		2^{-k}

The n_i series constitutes a geometric progression, the sum of which is $2^k - 1$. Thus, it is expected that all games but one will end within a series of k terms. It is also expected that each term in the series will in fact exist, ie α_i/N is small or equal to 0. The probability that all games will end by T_k thus equals zero. There is thus a series of k terms and single term exists beyond the k^{th} term. The total length of the series

is $k+1$ terms. Since N is an integer, only one game is expected to survive beyond the k^{th} . That single game can be any of the N games (i.e. it need not be the last game in the series of N games) but the probability decreases by $1/2$, (since an additional flip of the coin is required), for each term beyond T_k , that this surviving game is expected to end.

It should also be noted even if an enormous number of games is played, the series will be relatively short; certainly short enough to be summated manually, if necessary. The theory of harmonic series will not be necessary, in practice, to add the series even for exceptionally large values of N . For example if the game is played 2^{30} (1 073 741 824) times, the total length of the series is expected to consist of only $30 + 1 = 31$ terms, which can be summed manually without any difficulty.

Now in the Pasadena game, in order to arrive at a value of ≈ 4 in series (3) above the Pasadena series (1) must progress to the term with a value of $1/5279$ ie the series must have at least 5279 terms. This term is only expected to appear if the game is played at least $N = 2^{5279}$ times. If it is recalled that 2^{30} is in excess of a billion games, it is clear that 2^{5279} is a number which is so great that it is impossible to actually play (or want to play) the game that number of times. Even if it was possible to play the games that number of times, the total for the series (summed in the fashion set-out by the authors) is only 4.004351 (ie a long way from infinity)³. However if series (3) is examined it will be noted that only three negative terms ($-1/2$; $-1/4$ and $-1/6$) have been included. Since the series is naturally harmonic, the other negative terms do in fact exist and must be accounted for ie ($-1/8$ – $1/10$ – ...– $1/5278$). If these negative terms are summed their total is 3.311114 and the sum of the 5279 terms of the series becomes: $4.004351 - 3.311114 = 0.6932396$ which is approximately the value of $\ln 2$ (ie 0.693147106). One may reshuffle the

cards but when all is said and done, all the cards must still be summed, not only some of them.

A similar analysis can be carried out on series (4) which will produce a similar outcome.

3

Determining the expected value of the Pasadena game

The $E\{X\}$ of the Pasadena game is now determined.

Let $A(k)$ be the sum of the contributions of the first k terms and $B(k)$ the contribution of the final term to the $E\{X\}$ of the Pasadena game, then:

$$E\{X_k\} = A(k) + B(k)$$

4

Determining $A(k)$ for the Pasadena game

$E\{X_N\}$ is by definition $= \Sigma/N$ where S is the sum of the outcomes of $\Sigma n_i \cdot X_i$ and N the number of games which have been played. $E\{X_N\}$ is simply the average outcome of playing the game N times.

$$E\{X_N\} = S/N = n_1/N \cdot X_1 + n_2/N \cdot X_2 + n_3/N \cdot X_3 + \dots$$

For the Pasadena game $X_i = \$((-1)^{i-1} 2^i)/i$ and if $N = 2^k$ games are played then each of the terms in the above series contributes towards the final value of $E\{X_N\}$ as shown in Table 2 below:

Table 2

Term	T1	T2	T3	...	Tk	T(k+1)	T(k+2)	...	Total
n_i	2^{k-1}	2^{k-2}	2^{k-3}	...	2^{k-k}	1 or 0	1 or 0	1 or 0	N
X_i	$(-1)^{1-1} \frac{2^1}{1} = 2^1/1$	$(-1)^{2-1} \frac{2^2}{2} = -2^2/2$	$(-1)^{3-1} \frac{2^3}{3} = 2^3/3$...	$(-1)^{k-1} \frac{2^k}{k}$	$(-1)^k \frac{2^{k+1}}{(k+1)}$	$(-1)^{k+1} \frac{2^{k+2}}{(k+2)}$...	
$n_i/N \cdot X_i$	2^0	$-1/2$	$1/3$	—	$(-1)^{k-1} \frac{1}{k}$	$(-1)^k \frac{2^1}{(k+1)}$	$(-1)^{k+1} \frac{2^2}{(k+2)}$...	

From which it can be seen that for the first k terms of the series:

$$A(k) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots (-1)^{k-1}/k$$

for any finite k .

As pointed out the series is naturally harmonic with positive and negative terms appearing naturally in that order. The series is finite and does not need to be rearranged to arrive at a total and even if it is, if all the negative and positive terms are included one would arrive at the same answer.

The above series, if infinite in number, is the series for $\ln 2$. The series converges to $\ln 2 = 0.693147$ as the length of the series (i.e. number of games played) becomes increasingly large.

Thus $A(\infty) = \mu_{\infty}$ for the Pasadena game = $\ln 2 \approx 0.69$ i.e. as N or k becomes increasingly greater, μ_{∞} tends to $\ln 2$.

5

Determining $B(k)$, the contribution of the surviving game to the $E\{X_N\}$ of the Pasadena game

As indicated above, if $N=2^k$ games are played then $N-1$ games are expected to finish by the k^{th} term in the series. Thus only one game is expected to survive beyond this term. If $B(k)$ represents the value of the contribution to the $E\{X_N\}$ from this final term then the $E\{X_N\}$ of the Pasadena game is:

$$E\{X_N\} = A(k) + B(k)$$

Assuming $\alpha_i = 0$ for each of the k terms in the series, then it is expected that each term in the series will exist and the series will only terminate after the k^{th} term. There is thus a zero probability

that it will end before the k^{th} term and a $\frac{1}{2}$ (50 per cent) likelihood that all the games will end at the $k+1$ term, in which event this term will contribute $(-1)^k 2^1/(k+1)$ to the $E\{X_N\}$. There is a corresponding 50 per cent likelihood that all games will end beyond the $k+1$ term. Stating this another way one can be fifty percent confident that the series will terminate at the $k+1$ term.

There is a $\frac{1}{2^2}$ (25 per cent) likelihood that all the games will end at the $k+2$ term, in which event this term will contribute $(-1)^{k+1} 2^2/(k+2)$ to the $E\{X_N\}$. There is an accumulated likelihood of 75 per cent that all games will end on or before the $k+2$ term. Stating this another way one can be 75 per cent confident that the series will terminate by the $k+2$ term.

There is a $\frac{1}{2^3}$ (12.5 per cent) likelihood that all the games will end at the $k+3$ term, in which event this term will contribute $(-1)^{k+2} 2^3/(k+3)$ to the $E\{X_N\}$. There is an accumulated likelihood of 87.5 per cent that all games will end on or before the $k+3$ term. Stating this another way one can be 87.5 per cent confident that the series will terminate by the $k+3$ term.

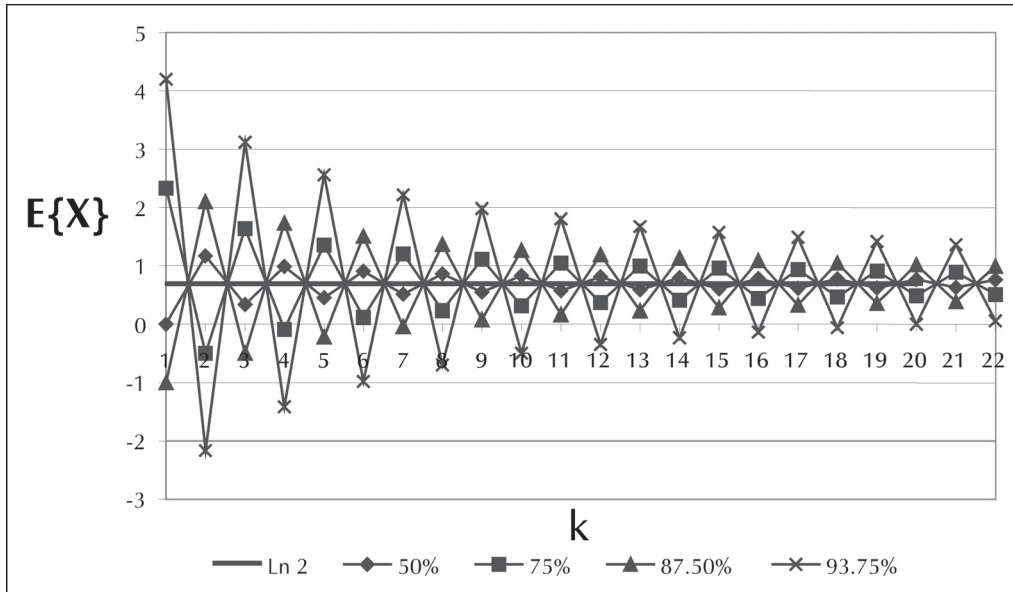
There is a $\frac{1}{2^4}$ (6.25 per cent) likelihood that all the games will end at the $k+4$ term, in which event this term will contribute $(-1)^{k+3} 2^4/(k+4)$ to the $E\{X_N\}$. There is an accumulated likelihood of 93.75 per cent that all games will end on or before the $k+4$ term. Stating this another way one can be 93.75 per cent confident that the series will terminate by the $k+4$ term.

And so on.

$E\{X_N\}$ can also be expressed as $\mu_g \pm \lambda(k)$. Where Σ is the difference between the actual outcome and μ_g . It will be noted that λ becomes increasingly small as N (and k) increases.

Figure 1

$E\{X\}$ of Pasadena games played 2^k times for confidence levels of 50 per cent; 75 per cent; 87.5 per cent and 93.75 per cent



The $E\{X_N\}$ s of Pasadena games which are played from 2^1 (2 games) to 2^{22} (4 194 312) games were calculated and the results are indicated in Table 3, for the above four confidence levels. The results are also indicated in Figure 1. It can be seen how the outcomes oscillate around $\ln 2$.

6 Simulation

The above theory can be applied as a practical problem and the results verified by computer simulation. Assume a casino owner wishes to offer the public the opportunity of playing the Pasadena game. The owner wants to know much he should ask gamblers to wager.

The solution to this problem will be something as follows:

It is estimated that 2^{19} (524 288) games will be played each month, or 6 291 456 million games a year. If the casino owner wishes to be 50 per

cent confident that he will make a profit, the $E\{X\} = A(k) + B(k) = 0.718 - 0.100$ or \$0.618 per game (see Table 3) above. If the casino owner wishes to be 93.75 per cent confident that he will make a profit, the $E\{X\} = A(k) + B(k) = 0.718 + 0.71$ or \$1.428 per game (see Table 3) above and so on.

Empirical results may also be obtained by simulating the games. The results of the simulations are indicated in Table 4 for a 12 months period, for confidence levels of 50 per cent (\$0.62 per game) and 93.75 per cent (\$1.41 per game). It will be seen that the simulated results indicate that the casino owner can expect to make a profit at both levels and not surprisingly he makes a greater profit at the 93.75 per cent confidence level. Because of the oscillating nature in two months (7 and 12) the casino owner, does not make a payment, but is paid by the gamblers.

Table 3
 $E\{X\}$ for $k=2$ to $k=22$ at various confidence levels

Confidence level														(2)
		50%			75%			87.50%			93.75%			
k	A(k)	B(k)	E{X}	?(k)	B(k)	E{X}	?(k)	B(k)	E{X}	?(k)	B(k)	E{X}	?(k)	
1	1.00	-1.00	0.00	-0.69	1.33	2.33	1.64	-2.00	-1.00	-1.69	3.20	4.20	3.51	0.69
2	0.50	0.67	1.17	0.47	-1.00	-0.50	-1.19	1.60	2.10	1.41	-2.67	-2.17	-2.86	0.69
3	0.83	-0.50	0.33	-0.36	0.80	1.63	0.94	-1.33	-0.50	-1.19	2.29	3.12	2.43	0.69
4	0.58	0.40	0.98	0.29	-0.67	-0.08	-0.78	1.14	1.73	1.03	-2.00	-1.42	-2.11	0.69
5	0.78	-0.33	0.45	-0.24	0.57	1.35	0.66	-1.00	-0.22	-0.91	1.78	2.56	1.87	0.69
6	0.62	0.29	0.90	0.21	-0.50	0.12	-0.58	0.89	1.51	0.81	-1.60	-0.98	-1.68	0.69
7	0.76	-0.25	0.51	-0.18	0.44	1.20	0.51	-0.80	-0.04	-0.73	1.45	2.21	1.52	0.69
8	0.63	0.22	0.86	0.16	-0.40	0.23	-0.46	0.73	1.36	0.67	-1.33	-0.70	-1.39	0.69
9	0.75	-0.20	0.55	-0.15	0.36	1.11	0.42	-0.67	0.08	-0.61	1.23	1.98	1.28	0.69
10	0.65	0.18	0.83	0.13	-0.33	0.31	-0.38	0.62	1.26	0.57	-1.14	-0.50	-1.19	0.69
11	0.74	-0.17	0.57	-0.12	0.31	1.04	0.35	-0.57	0.17	-0.53	1.07	1.80	1.11	0.69
12	0.65	0.15	0.81	0.11	-0.29	0.37	-0.33	0.53	1.19	0.49	-1.00	-0.35	-1.04	0.69
13	0.73	-0.14	0.59	-0.11	0.27	1.00	0.30	-0.50	0.23	-0.46	0.94	1.67	0.98	0.69
14	0.66	0.13	0.79	0.10	-0.25	0.41	-0.28	0.47	1.13	0.44	-0.89	-0.23	-0.92	0.69
15	0.73	-0.13	0.60	-0.09	0.24	0.96	0.27	-0.44	0.28	-0.41	0.84	1.57	0.87	0.69
16	0.66	0.12	0.78	0.09	-0.22	0.44	-0.25	0.42	1.08	0.39	-0.80	-0.14	-0.83	0.69
17	0.72	-0.11	0.61	-0.08	0.21	0.93	0.24	-0.40	0.32	-0.37	0.76	1.48	0.79	0.69
18	0.67	0.11	0.77	0.08	-0.20	0.47	-0.23	0.38	1.05	0.35	-0.73	-0.06	-0.75	0.69
19	0.72	-0.10	<u>0.62</u>	-0.07	0.19	0.91	0.22	-0.36	0.36	-0.34	0.70	<u>1.41</u>	0.72	0.69
20	0.67	0.10	0.76	0.07	-0.18	0.49	-0.21	0.35	1.02	0.32	-0.67	0.00	-0.69	0.69
21	0.72	-0.09	0.63	-0.07	0.17	0.89	0.20	-0.33	0.38	-0.31	0.64	1.36	0.66	0.69
22	0.67	0.09	0.76	0.06	-0.17	0.50	-0.19	0.32	0.99	0.30	-0.62	0.06	-0.64	0.69

Table 4
Results of simulation of the Pasadena game

Month	Games played	Income		Payments (determined from simulating the game)	Profit	
		50%	93.5%		50%	93.5%
1	524,288	\$325,058.56	\$739,246.08	\$159,732.29	\$165,326.27	\$579,513.79
2	524,288	\$325,058.56	\$739,246.08	\$364,323.23	-\$39,264.67	\$374,922.85
3	524,288	\$325,058.56	\$739,246.08	\$321,081.83	\$3,976.73	\$418,164.25
4	524,288	\$325,058.56	\$739,246.08	\$313,134.42	\$11,924.14	\$426,111.66
5	524,288	\$325,058.56	\$739,246.08	\$710,653.92	-\$385,595.36	\$28,592.16
6	524,288	\$325,058.56	\$739,246.08	\$394,595.07	-\$69,536.51	\$344,651.01
7	524,288	\$325,058.56	\$739,246.08	-\$293,183.71	\$618,242.27	\$1,032,429.79
8	524,288	\$325,058.56	\$739,246.08	\$455,697.66	-\$130,639.10	\$283,548.42
9	524,288	\$325,058.56	\$739,246.08	\$349,582.40	-\$24,523.84	\$389,663.68
10	524,288	\$325,058.56	\$739,246.08	\$343,957.75	-\$18,899.19	\$395,288.33
11	524,288	\$325,058.56	\$739,246.08	\$334,912.83	-\$9,854.27	\$404,333.25
12	524,288	\$325,058.56	\$739,246.08	-\$334,728.34	\$659,786.90	\$1,073,974.42
		\$3,900,702.72	\$8,870,952.96	\$3,119,759.36	\$780,943.36	\$5,751,193.60

7 Conclusion

It is thus clear that the so-called Pasadena game does not lead to a paradox in the St Petersburg sense of a paradox. Further it does not produce an anomaly in the field of decision theory. Empirical results are in line with that predicted by the expected value criterion.

Endnotes

- 1 Vivian (2003) argued that the traditional derivation of the expected value $E\{X\} = 8$, of the St Petersburg game is correct only as a special case ie where an infinite number of games are played. Generally $E\{X\}$ is a function of the number of games played N and is $E\{X\} = k/2 + \lambda$, when the game is played $N=2^k$ times and $\tilde{\epsilon}$ chosen to give the desired degree of confidence. Vivian (2004) validated empirically the correctness of the equation.
- 2 The statement was made by Bernoulli, however it is of course not possible, as Menger (*infra*) pointed out, for the series to extend to infinity,

since in this event $N = 8$ in which case $N/2$ or $N/4$ etc becomes meaningless.

- 3 The total was derived from setting up the series and summing it on an Excel spreadsheet.

References

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