Revised 004-8 Considering the Harmonic Sequence "Paradox"

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Abstract

Blavatskyy (2006) formulated a game of chance based on the harmonic series which, he suggests, leads to a St Petersburg type of paradox. In view of the importance of the St Petersburg game in decision theory, any game which leads to a St Petersburg game type paradox is of interest. Blavatskyy's game is re-examined in this article to conclude that it does not lead to a St Petersburg type paradox.

Keywords: St Petersburg paradox; harmonic series; harmonic series paradoxes; decision theory and games of chance; decision theory paradoxes; expected values.

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1 Introduction

Blavatskyy (2006) formulated a game based on the well-known harmonic series which he argues leads to a decision theory paradox which "strengthens the original and super St Petersburg paradox." He appropriately named his perceived St Petersburg type paradox as the Harmonic Sequence Paradox. His game and suggested paradox are examined herein and it is concluded that it does not lead to a St Petersburg type of paradox. Blavatskyy is not the first to attempt to devise a game based on the harmonic series and to suggest that it leads to a St Petersburg type of paradox.[2]

Decision theory involving risk is often studied against the background of games of chance. A number of rules have been developed to indicate the fair amount that a person, the proverbial Paul, should be prepared to pay the casino operator (the proverbial Peter), to take part in a specific game.[3] The first of these rules is the expected monetary value (EMV) rule. Credit is usually given to De Fermat (1601-1665) and especially Pascal (1623-1662) for the formulation of this rule (Samuelson 1977:37). This, as a rule of thumb, suggests that Paul should be prepared to pay an amount approximately equal to the calculated value of the expected value of the game. Games of chance can also be and usually are played repetitively. Usually a measure of consistency exists between the expected value of a game and the amount that Paul is, in fact, prepared to wager. In this instance there is no paradox. Sometimes authors believe they have formulated a game which leads to a decision theory paradox. This would happen for example where the calculated expected value of the game is very large (even in theory infinite) but empirical evidence or common sense indicates that Paul is prepared to pay, comparatively speaking, only a very modest sum to participate in the game. Therein lies the paradox; theory and reality appear to diverge or as Todhunter (1865:220), setting out the history of the paradox succinctly put it, "[t]he paradox then is that the mathematical theory is apparently directly opposed to the dictates of common sense". Should this be the case, the expected value rule appears to fail. Should any game be formulated which does indeed produce a paradox of this nature, this has serious implications for the validity of the expected value rule, indicating that it is an unreliable decision rule and then an alternative more reliable decision rule should be sought. The most famous game of this sort is the so-called St Petersburg game leading to the St Petersburg paradox[4] out of which the second decision rule (or theory), the expected utility value (EUV) rule, evolved. This second rule is evoked where it is demonstrated that a paradox exists. If no paradox is shown to exist, in the Todhunter sense, it is not necessary to evoke the second rule.

Detailed examinations of games which are thought to produce these paradoxes demonstrate that these games in fact do not do so. Usually it is the expected value of the game which has been incorrectly calculated (as in the case of the St Petersburg game[5]) or the game is incorrectly formulated or misconstrued as a game. More recently authors simply do not state what they mean by a *paradox.* With this background Blavatskyy's harmonic sequence "game" and "paradox" is re-examined.

2
Blavatskyy's "game"

The point of departure in the analysis of any game is to understand what the formulated game is, what expected outcomes should appear when the game is played and what happens when the game is played repeatedly. So to start, Blavatskyy's game must be identified which he formulated as follows:

> Consider an urn that initially contains one white and one black ball. An individual draws one ball from this urn and receives one dollar (nothing) should the ball be white (black). Whatever the drawn ball happens to be, it is subsequently put back into the urn. Additionally, one additional black ball is added to the urn. The individual then draws one ball again and the cycle continues *ad infinitum*. At each iteration, the drawn white ball pays off one dollar and the number of black balls is increased by one. What is a maximum price that a rational individual is willing to pay for participation in this lottery L?

(Italics in the original)

According to Blavatskyy's formulation, his "game" is one of infinite duration; or in his words it continues *ad infinitum*. Clearly this as a game is misconstrued. A "game" of infinite duration is neither conceptually sound nor practically possible. Since this "game" has no end, it can never be completed and hence never repeated or played repetitively. Games of "infinite" duration can either tend to a converging value or diverge. If the game produces a diverging outcome then the longer the game is played the greater is the outcome, albeit the outcome may diverge at a very slow rate. The diverging "game" thus has no single outcome. Thus for example, if a coin is flipped **n** times and each time a head appears, a dollar is paid out, the expected value of a game of **n** flips is $1/2 **n**. Clearly, the greater the number of flips, the greater is the expected outcome. There is no single defined outcome for this "game". That playing a "game" of "infinite" duration can produce an "infinite" payout is of course not surprising. Games of finite duration, with finite outcomes, repeated "infinitely", can also produce an "infinite" aggregate outcome. Virtually any game of chance, repeated a very large number of times will produce a divergent aggregate outcome. This does not mean that Paul will become a billionaire from playing any of these games since the casino operator will charge Paul to play the game. To play forever, will cost Paul billions which would wipe out expected payouts. In fact, in the long run as Adam Smith (1776:Bk1.ChX.PtI) pointed out:

> [t]here is not, however, a more certain proposition in mathematics, than the more [lottery] tickets you adventure upon, the more likely you are to be a loser. Adventure upon all the tickets in the lottery, and you lose for certain; and the greater the number of your tickets the nearer you approach this certainty.

Thus despite the large aggregate payout, Paul is bound to be the loser, the longer he plays. Not even the St Petersburg game is formulated as a game of infinite duration. The St Petersburg game consists of flipping a coin until a head appears. The game then ceases. This game can of course then be repeated, conceptually an infinite, or at least a large number of times, if, Paul so chooses. The duration of any individual St Petersburg game is generally very short in duration. The problem of the St Petersburg game is to determine the expected value of a game of a finite duration, but the point of termination is unknown.

Since the eternal flipping of a coin produces a divergent outcome it should not be formulated as a game. Rather it is the single flip of the coin which is defined as the game. This game can then be repeated and how the expected values of a single flip of the coin (or the average outcome) changes when the game is repeated a number of times can be studied (Vivian, 2003a).

Blavatskyy's 'game' is thus misconstrued as a game. In order to make some sense of Blavatskyy's game, it has to be confined to a finite number say **n** extractions from the urn. The expected payout (paying $1 every time a white ball is drawn) of Blavatskyy's game if played for duration of **n** extractions approximates to:

$S_n \approx 1/2 + 1/3 + 1/4 \ldots + 1/(n+1)$

This is of course a subset of the well-known harmonic series[6] which is now known to be a divergent series.[7] The eternal extractions from the urn clearly cannot be defined as a game.

3 Does Blavatskyy's "game" produce a paradox?

If Blavatskyy's game is played for **n** iterations does it produce a paradox? As indicated the expected value of the harmonic series game played **n** iterations is:

$S_n \approx 1/2 + 1/3 + 1/4 \ldots + 1/(n+1)$
or
$S_n \approx \ln \mathbf{n} - 0.4228$[8]

if $n = e^k$, then

$S_n \approx k - 0.4228$

From which it is clear that it is a simple matter to determine the EMV of a game consisting of n iterations. At this juncture it is appropriate to recall that the St Petersburg game is thought to produce a paradox because of the great divergence between the theoretical prediction and the empirical evidence of the amount Paul is prepared to pay. The traditional determination of the EMV of the St Petersburg game is infinite but empricial evidence indicates Paul is prepared to pay only a very modest sum of about $13 to play the game; hence the paradox. No such paradox exists with regard to Blavatskyy's game. Even for a game consisting of a very large number of extractions, S_n is still very modest and can be determined with a high degree of confidence from the above equation. According to Blavatskyy (2006:221) (presumably shown empirically) participants to the game will only offer a very modest amount to participate. Theory and reality thus converge and no paradox in the Todhunter sense exists. Oddly despite indicating that his game produces a paradox, Blavatskyy does not clearly indicate what he considered to be the paradox.

4 Who is Paul?

As indicated historically, a paradox is said to exist because of the mismatch of Paul's assessment of the value of a game and the calculated value. As will be argued, the mismatch depends to a large measure on who Paul is[9]. Although it is accepted in the case of the St Petersburg game, Paul will be prepared to offer only a modest amount to play the game, little is said about how he arrives at his conclusion. The most common mechanism to arrive at a value is to explain the game to an audience and to ask them to write a number on a piece of paper and then, after excluding outliers, to take the average of the numbers provided. A conceptual advancement on this traditional method is now suggested. Consider three different categories into which Paul may fall. Paul may be a novice (an uninitiated), a mathematician, or an observer. Depending on which of the three he is, he may have a different answer to the question as to the value of any game. To understand this, it is again necessary to take a step back in time.

One common view (not necessarily the correct view) of the origins of probability theory is that it evolved to solve gambling problems, specifically those posed by Chevalier de Méré (1607-1684) to Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662) (Maistrov 1974). One such problem was to determine the probability of a six not turning up when a die is thrown four times in a row, (or four dice are thrown once) (Maistrov 1974:41). If the six does turn-up Paul wins. What De Méré needed was a mathematician to calculate the probability so that he could know with confidence that if he offered this game, that he would win, in the long run. At the time no-one knew how to calculate this probability and its value could not be merely guessed with confidence. The probability was calculated by the mathematicians of the day to be 625/1296 or 0.4823; just short of 50 per cent. This probability marginally favoured De Méré. Since the average gambler who is offered this game (at the time) could not work out this probability, De Méré reasoned that this marginal advantage would make him very wealthy in the long run. From this background it can be accepted as a point of departure that at least since the mid 1650s it has been known and historically accepted fact that the uninitiated (or novice) gambler is incapable of correctly estimating the probability or expected value of a complex game.

On the other hand, since the mid 1650s, equally, it also can be accepted that mathematicians may indeed be capable of correctly working out expected values of complex games. Mathematicians can however, and often do, make mistakes.

Since mathematicians can and do make mistakes a third procedure to determine an amount Paul should be prepared to wager is suggested and that is to observe the outcomes when the game is played repeatedly. Since the game can be played and outcomes recorded, the observer, even if unskilled in mathematics, would be in a better position to make an informed decision on the amount to wager. Thus for example if the St Petersburg game is played a billion times it will be observed that the expected value of all of these games is a mere $15 per game played (Vivian 2003). Paul having observed this will be very irrational to wager say a large sum, say $200 000 per game. In the age of the computer, games can be simulated to determine the expected value when played a large number of times.

4.1 Paul the novice or uninitiated

With this the elucidation of the historical position, Blavatskyy's paradox is returned to. Blavatskyy gets close to understanding that his game does not produce a paradox when he states, "The proposed paradox may be refuted on practical grounds. Since it employs a lottery with [a] logarithmically growing mean, about 10^{43} iterations are required to guarantee at least a modest expected outcome of 100 dollars." If Blavatskyy's game consists of 10^{43} iterations, then the above formula produces an expected value of $98.59. A more accurate statement of the expected value is $98.59 ± λ where λ represents an amount chosen which is commensurate with the level of confidence in that outcome (Vivian 2003a). So if Paul selects a value of say $100 to participate in the game there is no paradox; theory and practice are in agreement. Blavatskyy, unfortunately, does not provide any empirical evidence as to the amount Paul will offer to participate in a game of 10^{43} extractions but he immediately goes on to suggest a game with a different number of iterations, 10^{15} which he suggests can be played using a high speed computer in a period of 5 minutes. Using the above formula the expected value of a game consisting of 10^{15} extractions is $34.54 ± λ. If the desired confidence level is ignored (as is usually the case) the game will yield an expected value of $34.54. Blavatskyy notes however, that Paul's willingness to pay (WTP) is less than $10. He does not provide any evidence in support of the $10 or indicate how this figure is arrived at. A novice may well suggest an amount of $10 as the maximum amount they are prepared to pay, but neither a mathematician nor anyone who has observed the outcome of a game consisting of 10^{15} extractions as obtained by simulating the game using a high speed computer would do so.

Now what kind of Paul will offer only $10 when the calculated and observed expected value is of the order of $34.54? He must be a novice. He is not a mathematician. If for example one asks a number of students who have no knowledge of mathematics and explain to them, only once, the harmonic series game and then say to them, "The game is now going to be played once,

10^{15} times. Write on a piece of a paper what you believe the expected value of the game will be?" There can be little doubt that these totally uninitiated students will not accurately guess the correct answer. This simply means that inexperienced persons do not accurately predict expected outcomes of complex games. As pointed out above this has been accepted since the 1650s. If Blavatskyy had stated his problem as, "Do uninitiated persons accurately predict expected values of complex games?" Then the above experiment based on his game will provide the appropriate and well-known answer; no they do not. This however is neither a paradox or new. It is not a contradiction in decision theory to hold that persons cannot correctly guess expected values of complex games. This is where decision theory started in the mid 1650s when De Méré set about involving mathematicians in solving complex gambling problems. The fact that in Blavatskyy's game persons are unable to correctly estimate the expected value of this complex game does not demonstrate a paradox but confirms that which has been known for a long time; the uninitiated cannot accurately estimate the probabilities (or expected values) of complex games of chance. After all, it took some of the world's most famous mathematicians to prove the harmonic series does indeed diverge.

4.2 Paul the mathematician

Generally, when attempting to design a decision paradox it should be accepted that Paul is a mathematician (or has hired one) and not a mere novice. To accept otherwise is simply to continually re-prove that novices cannot correctly estimate expected outcomes of complex games of chance. Paul is capable of determining, mathematically, correctly the expected value of the proposed game. In this case we can accept that Paul as a mathematician (or the one he has hired) can calculate the expected value of the game and will arrive at the above formula. In this event he will calculate the breakeven expected value of \$36 from playing the game with 10^{15} iterations. In history mathematicians have not always been a good option. They have been known to make mistakes. When De Méré's game of no sixes ceased to produce an income he invented another even more complex game. This time however he (or his hired mathematicians) made a mistake in the calculation of the expected value. Unbeknown to him the marginal bias was against him. The more he played the more he lost. He put all his faith in the incorrect calculation. In the end "he ruined himself and ultimately he ended up impoverished" (Maistrov 1974:41).

4.3 Paul the observer

If Paul exercises a bit of common sense he would go to a casino where harmonic sequence games are played and observe the games for some time. Each game consisting of 10^{15} extractions, after all, using a high speed computer, takes only 5 minutes to complete. After observing this game being played for a couple of hours and noting that almost invariably the game produces an outcome of the order \$36, if he is then asked to predict the outcome there can be little doubt he will select a figure of the order of \$36 and not the much smaller figure of \$10 which Blavatskyy puts forward. It is the failure to understand the importance of experience or observed results which, incidentally, legend has it, that Chevalier de Méré did not become wealthy from his game of no sixes, as Maistrov records. Players soon discovered that by playing the game long enough they lost money. The game was not fair to them and they stopped playing. Observation or experience soon reveals, as Adam Smith noted the correctness of expected outcomes of games. In a sense a casino operator is an observer. In operating the casino and maintaining a set of accounts the operator will get a good feel for the expected values of specific games, or from Adam Smith's observation the operator will know if he is making or losing money.

It can be concluded it is only a novice who will estimate the expected value of Blavatskyy's game consisting of 10^{15} iterations to be \$10, a figure easily refuted by calculation or observation. So the question in deciding if a paradox exists should be: do the calculated and empirical determined values of the expected value inexplicitly differ? The empirical value should be determined by simulation, not from asking the uninitiated to guess the value.[10] The question cannot be: do uninitiated persons guess a value different from the calculated value?

5 Conclusion

Blavatskyy attempted to formulate a game based on the harmonic series to produce a decision theory paradox akin to the St Petersburg paradox. An examination of his attempt however reveals that his "game" does not do so. There is no divergence between the value determined by the application of the expected monetary value rule and what a rational, experienced person will offer to pay to play the game. This being so, there is no need to evoke the expected utility value theory to explain the choice made by Paul. Blavatskyy's attempt does however reveal a methodological problem with attempts to produce a St Petersburg type of paradox and that is, the failure to clearly define the problem being addressed. To constitute a paradox the question should be: does the objectively determined empirical value of the expected outcome of a game differ significantly and inexplicably from the mathematically determined value? If it does, history suggests that the mathematical determination of the expected value be checked.

Endnotes

1. This article has benefited from the insightful comments of two anonymous referees. The usual disclaimers apply.
2. This was recently done by Nover et al. (2004) who suggested a paradox from the alternating harmonic series. For a demonstration that their suggested game does not lead to a paradox of the St Petersburg paradox type see Vivian (2006). In examining Nover et al. (2004) games, Fine (2008:467) has also concluded (for different reasons), "… that there is no failure of standard utility theory in assessing these gambles and no paradox."
3. Reference to Peter and Paul comes from Bernoulli (1954/1738).
4. The St Petersburg game was first set-out by Nicolas Bernoulli in a letter to De Montmort (1678-1719) in 1713.
5. That the St Petersburg game does not produce a paradox see Vivian (2003).
6. More fully the harmonic series is 1+ 1/2 + 1/3 + 1/4 ... + 1/n.
7. The rather neat proof that the harmonic series is divergent was first set-out by Nicole d'Oresme (ca 1323-1382) which proof seemed to be mislaid for several centuries and three further proofs were advanced by Pietro Mengoli (1647), Johann Bernoulli (1687) and shortly thereafter by Jakob Bernoulli. For a recent short history of the harmonic series see Derbyshire (2003).
8. Blavatskyy (2006:222).
9. Traditionally three different categories of decision makers can be identified: risk takers, persons who are risk averse and those who are risk neutral.
10. For a simulated demonstration that the expected value of the St Petersburg game can be determined with a high degree of confidence see Vivian (2004).

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